Lecture 11

Transmission Lines

Transmission lines represent one of the most important electromagnetic technologies. The reason being that they can be described by simple theory, similar to circuit theory. As such, the theory is within the grasp of most practicing electrical engineers. Moreover, transmission line theory fills the gap in the physics of circuit theory: Circuit theory alone cannot describe wave phenomena, but when circuit theory is augmented with transmission line theory, wave phenomena with its corresponding wave physics start to emerge.

Even though circuit theory has played an indispensable role in the development of the computer chip industry, eventually, circuit theory has to be embellished by transmission line theory, so that high-speed circuits can be designed. Retardation effect, which causes time delay, clock skew, and phase shift, can be modeled simply using transmission lines. Nowadays, commercial circuit solver software such as SPICE[1] have the capability of including transmission line as an element in modeling.

[1]This is an acronym for a package "simulation program with integrated circuit emphasis" that came out of U. Cal., Berkeley [87].

11.1 Transmission Line Theory

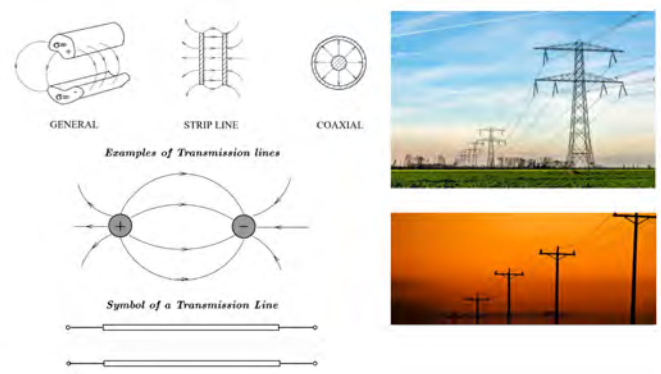

Figure 11.1: Various kinds of transmission lines. Schematically, all of them can be modeled by two parallel wires. On the right are pictures of a power transmission line, and a telephone lineḊue to technology advancement, telephone lines are a rare sight now (courtesy of Lister-Communications and Istockphoto.com).

Transmission lines were the first electromagnetic waveguides ever invented. The were driven by the needs in telegraphy technology. It is best to introduce transmission line theory from the viewpoint of circuit theory, which is elegant and one of the simplest theories of electrical engineering. This theory is also discussed in many textbooks and lecture notes. Transmission lines are so important in modern day electromagnetic engineering, that most engineering electromagnetics textbooks would be incomplete without introducing the topics related to them [32, 33, 45, 51, 55, 66, 81, 85, 88, 89].

Circuit theory is robust and is not sensitive to the detail shapes of the components involved such as capacitors or inductors. Moreover, many transmission line problems cannot be analyzed simply when the full form of Maxwell's equations is used,[2] but approximate solutions can be obtained using circuit theory. We shall show later that circuit theory is an approximation of electromagnetic field theory when the wavelength is very long: the longer the wavelength, the better is the approximation [51]. Hence, in long-wavelength limit, transmission line theory can be approximated by circuit theory.

Examples of transmission lines are shown in Figure 11.1. The symbol for a transmission line is usually represented by two pieces of parallel wires, but in practice, these wires need not be parallel as shown in Figure 11.2.

[2] Usually called full-wave analysis.

Figure 11.2: A twisted pair transmission line where the two wires are not parallel to each other (courtesy of slides by A. Wadhwa, A.L. Dal, N. Malhotra [90].)

Circuit theory also explains why waveguides can be made sloppily when wavelength is long or the frequency low. For instance, in the long-wavelength limit, we can make twisted-pair waveguides with abandon, and they still work well (see Figure 11.2). Hence, it is simplest to first explain the propagation of electromagnetic signal on a transmission line using circuit analysis.

11.1.1 Time-Domain Analysis

We will start with performing the time-domain analysis of a simple, infinitely long transmission line. Remember that two pieces of metal can accumulate attractive positive and negative charges between them, giving rise to electric fields that start with positive charges and end with negative charges. The stored energy in the electric field gives rise to capacitive effect in the line which can be modeled by capacitances. Moreover, a piece of wire carrying a current generates a magnetic field, and hence, yields stored energy in the magnetic field. The stored magnetic field energy gives rise to inductive effect in the line which can be modeled by inductances. These stored energies are the sources of the capacitive and inductive effects.

But these capacitive and inductive effects are distributed over the spatial dimension of the transmission line. Therefore, it is helpful to think of the two pieces of metal as consisting of small segments of metal concatenated together. Each of these segments will have a small inductance, as well as a small capacitive coupling between them. Hence, we can model two pieces of metal with a distributed lumped element model as shown in Figure 11.3. For simplicity, we assume the other conductor to be a ground plane, so that it need not be approximated with lumped elements.

In the transmission line, the voltage $V(z,t)$ and the current $I(z,t)$ are functions of both space z and time t, but we will model the space variation of the voltage and current with discrete step approximations. The voltage varies from node to node while the current varies from branch to branch of the lumped-element model.

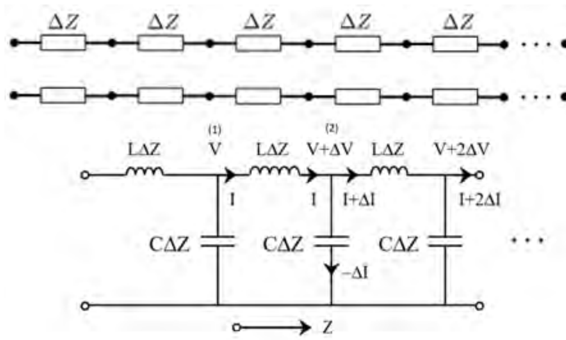

Figure 11.3: A long transmission line can be replaced by a concatenation of many short transmission lines. For each pair of short wires, there are capacitive coupling between them. Furtheremore, when current flows in the wire, magnetic field is generated making them behave like an inductor. Therefore, the transmission line can be replaced by a lumped-element approximation as shown. The lumped elements have inductances given by $L\Delta z$ and capacitances given by $C\Delta z$, distributed over the line. Hence, this is also known as the distributive model of the transmission line.

Telegrapher's Equations

First, we recall that the V-I relation of an inductor is

$$V_0 = L_0 \frac{dI_0}{dt} \tag{11.1.1}$$

where L_0 is the inductor, V_0 is the time-varying voltage drop across the inductor, and I_0 is the current through the inductor. Then using this relation between nodes 1 and 2 in Figure 11.3, we have

$$V - (V + \Delta V) = L\Delta z \frac{\partial I}{\partial t} \tag{11.1.2}$$

The left-hand side is the voltage drop across then series inductor, while the right-hand side follows from the aforementioned V-I relation of an inductor, but we have replaced $L_0 = L\Delta z$. Here, L is the inductance per unit length (line inductance) of the transmission line. And $L\Delta z$ is the incremental inductance due to the small segment of metal of length Δz. In the above, we assume that $V = V(z,t)$ and $I = I(z,t)$, so that time derivative is replaced by partial time derivative. Then the above (11.1.2) can be simplified to

$$\Delta V = -L\Delta z \frac{\partial I}{\partial t} \tag{11.1.3}$$

where ΔV is the incremental voltage drop between the two nodes 1 and 2.

Next, we make use of the V-I relation for a capacitor, which is

$$I_0 = C_0 \frac{dV_0}{dt} \tag{11.1.4}$$

where C_0 is the capacitor, I_0 is the current through the capacitor, and V_0 is a time-varying voltage drop across the capacitor. Thus, applying this relation at node 2 in Figure 11.3 gives the incremental shunt current to be

$$-\Delta I = C\Delta z \frac{\partial}{\partial t}(V + \Delta V) \approx C\Delta z \frac{\partial V}{\partial t} \tag{11.1.5}$$

where C is the capacitance per unit length, and $C\Delta z$ is the incremental capacitance between the small piece of metal and the ground plane. In the above, we have used Kirchhoff current law to surmise that the current through the shunt capacitor is $-\Delta I$, where $\Delta I = I(z + \Delta z, t) - I(z,t)$. In the last approximation in (11.1.5), we have dropped a term involving the product of Δz and ΔV, since it will be very small or second order in magnitude.

In the limit when $\Delta z \to 0$, one gets from (11.1.3) and (11.1.5) that

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t} \tag{11.1.6}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t} \tag{11.1.7}$$

The above are the *telegrapher's equations.*[3] They are two coupled first-order equations, and can be converted into second-order equations easily by eliminating one of the two unknowns. Therefore,

$$\frac{\partial^2 V}{\partial z^2} - LC\frac{\partial^2 V}{\partial t^2} = 0 \tag{11.1.8}$$

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = 0 \tag{11.1.9}$$

The above are wave equations that we have previously studied, where the velocity of the wave is given by

$$v = \frac{1}{\sqrt{LC}} \tag{11.1.10}$$

Furthermore, if we assume that

$$V(z,t) = V_0 f_+(z - vt), \qquad I(z,t) = I_0 f_+(z - vt) \tag{11.1.11}$$

corresponding to a right-traveling wave, they can be verified to satisfy (11.1.6) and (11.1.7) as well as (11.1.8) and (11.1.9) by back substitution.

Consequently, we can easily show that for the right-traveling wave,

$$\frac{V(z,t)}{I(z,t)} = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = Z_0 \tag{11.1.12}$$

where Z_0 is the characteristic impedance of the transmission line. The above ratio is only true for one-way traveling wave, in this case, one that propagates in the $+z$ direction.

[3]They can be thought of as the distillation of the Faraday's law and Ampere's law from Maxwell's equations without the source term. Their simplicity gives them an important role in engineering electromagnetics.

For a wave that travels in the negative z direction, we can let,

$$V(z,t) = V_0 f_-(z+vt), \qquad I(z,t) = I_0 f_-(z+vt) \tag{11.1.13}$$

one can easily show by the same token that

$$\frac{V(z,t)}{I(z,t)} = \frac{V_0}{I_0} = -\sqrt{\frac{L}{C}} = -Z_0 \tag{11.1.14}$$

Time-domain analysis is very useful for transient analysis of transmission lines, especially when nonlinear elements are coupled to the transmission line.[4] Another major strength of transmission line model is that it is a simple way to introduce time-delay (also called retardation) in a simple circuit model.[5] As we saw when we studied the solution to the wave equation: solutions at different times are just the time-delayed version of the original solution.

Time Delay and Velocity of Light

Time delay is a wave propagation effect, and it is harder to incorporate into circuit theory or a pure circuit model consisting of R, L, and C only. In circuit theory, where the wavelength is assumed very long, Laplace's equation is usually solved, which is equivalent to Helmholtz equation with infinite wave velocity, namely,

$$\lim_{c\to\infty} \nabla^2\Phi(\mathbf{r}) + \frac{\omega^2}{c^2}\Phi(\mathbf{r}) = 0 \quad \Longrightarrow \quad \nabla^2\Phi(\mathbf{r}) = 0 \tag{11.1.15}$$

From the above, we see that Helmholtz equation becomes Laplace's equation when the velocity of light c is infinite. Hence, events in Laplace's equation happen instantaneously. In other words, circuit theory, where Laplace's equation is usually solved, assumes that the velocity of the wave is infinite, and there is no retardation effect. This is only true or a good approximation when the size of the structure is small compared to wavelength.

11.1.2 Frequency-Domain Analysis—the Power of Phasor Technique Again!

As we have seen in previous lectures, the frequency-domain analysis greatly simplifies the analysis of many complicated phenomena. This was especially true in our analysis of conductive media, and frequency dispersive media as in the Drude-Lorentz-Sommerfeld model. As such, frequency domain analysis is very popular as it makes the transmission line equations very simple–one just replace $\partial/\partial t \rightarrow j\omega$. Moreover, generalization to a lossy system is quite straight forward. Furthermore, for linear time invariant systems, the time-domain signals can be obtained from the frequency-domain data by performing a Fourier inverse transform since phasors and Fourier transforms of a time variable are just related to each other by a constant.

[4] Remember that we can only use frequency domain technique or Fourier transform for linear time-invariant systems.

[5] By a simple circuit model, we mean a model that has lumped elements such as R, L, and C as well as a transmission line element.

The telegrapher's equations (11.1.6) and (11.1.7) then in frequency domain become

$$\frac{d}{dz}V(z,\omega) = -j\omega L I(z,\omega) \tag{11.1.16}$$

$$\frac{d}{dz}I(z,\omega) = -j\omega C V(z,\omega) \tag{11.1.17}$$

The above gives the notion that the change in the voltage $V(z,\omega)$ on a transmission line is proportional to the line impedance $j\omega L$ times the current $I(z,\omega)$. Similar notion can be said of the second equation above.

The corresponding Helmholtz equations are then

$$\frac{d^2V}{dz^2} + \omega^2 LCV = 0 \tag{11.1.18}$$

$$\frac{d^2I}{dz^2} + \omega^2 LCI = 0 \tag{11.1.19}$$

The above are second ordinary differential equations, and the general solutions to the above are

$$V(z) = V_+e^{-j\beta z} + V_-e^{j\beta z} \tag{11.1.20}$$

$$I(z) = I_+e^{-j\beta z} + I_-e^{j\beta z} \tag{11.1.21}$$

where $\beta = \omega\sqrt{LC}$. This is similar to what we have seen previously for plane waves in the one-dimensional wave equation in free space, where

$$E_x(z) = E_{0+}e^{-jk_0 z} + E_{0-}e^{jk_0 z} \tag{11.1.22}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. We see much similarity between (11.1.20), (11.1.21), and (11.1.22).

To see the solution in the time domain, we let the phasor $V_\pm = |V_\pm|e^{j\phi_\pm}$ in (11.1.20), and the voltage signal above can then be converted back to the time domain using the key formula in phasor technique as

$$V(z,t) = \Re e\{V(z,\omega)e^{j\omega t}\} \tag{11.1.23}$$

$$= |V_+|\cos(\omega t - \beta z + \phi_+) + |V_-|\cos(\omega t + \beta z + \phi_-) \tag{11.1.24}$$

As can be seen, the first term corresponds to a right-traveling wave, while the second term is a left-traveling wave.

Furthermore, if we assume only a one-way traveling wave to the right by letting $V_- = I_- = 0$, then it can be shown that, for a right-traveling wave, using (11.1.16) or (11.1.17)

$$\frac{V(z)}{I(z)} = \frac{V_+}{I_+} = \sqrt{\frac{L}{C}} = Z_0 \tag{11.1.25}$$

where Z_0 is the characteristic impedance. Since Z_0 is real, it implies that the phasors[6] $V(z)$ and $I(z)$ are in phase.

[6] We will neglect to denote phasors by under-tilde, as they are implied by the context.

Similarly, applying the same process for a left-traveling wave only, by letting $V_+ = I_+ = 0$, then

$$\frac{V(z)}{I(z)} = \frac{V_-}{I_-} = -\sqrt{\frac{L}{C}} = -Z_0 \tag{11.1.26}$$

In other words, for the left-traveling waves, the voltage and current are 180 ° out of phase.

11.2 Lossy Transmission Line

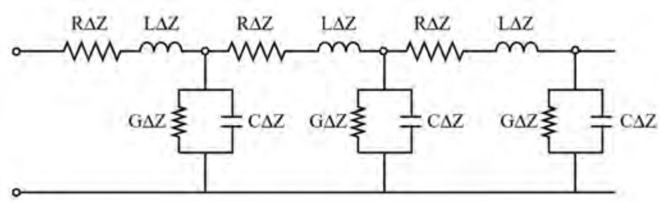

Figure 11.4: In a lossy transmission line, series resistance can be added to the series inductance, and the shunt conductance can be added to the shun susceptance of the capacitor. However, this problem is homomorphic to the lossless case in the frequency domain.

The phasor technique is empowered by that the algebra for complex numbers is the same as that of real numbers. Second, resistors and conductances are replaced by impedances and admittances in the frequency domain, making the solution to a network of impedances and admittances analogous to the network of resistances and conductances. The power of frequency domain analysis is revealed in the study of lossy transmission lines. The previous analysis, which is valid for lossless transmission line, can be easily generalized to the lossy case in the frequency domain. In using frequency domain and phasor technique, impedances will become complex numbers as shall be shown.

To include loss, we use the lumped-element model as shown in Figure 11.4. One thing to note is that $j\omega L$ is actually the series line impedance of the lossless transmission line, while $j\omega C$ is the shunt line admittance of the same line. First, we can rewrite the expressions for the telegrapher's equations in (11.1.16) and (11.1.17) in terms of series line impedance and shunt line admittance to arrive at

$$\frac{d}{dz}V = -ZI \tag{11.2.1}$$

$$\frac{d}{dz}I = -YV \tag{11.2.2}$$

where $Z = j\omega L$ and $Y = j\omega C$. The above can be easily generalized to the lossy case as shall be shown.

The geometry in Figure 11.4 is topologically similar to, or homomorphic[7] to the lossless case in Figure 11.3. Hence, when lossy elements are added in the geometry, we can surmise that the corresponding telegrapher's equations are similar to those above. But to include loss, we need only to generalize the series line impedance and shunt admittance from the lossless case to lossy case as follows:

$$Z = j\omega L \rightarrow Z = j\omega L + R \tag{11.2.3}$$

$$Y = j\omega C \rightarrow Y = j\omega C + G \tag{11.2.4}$$

where R is the series line resistance, and G is the shunt line conductance, and now Z and Y are the series impedance and shunt admittance, (which are complex numbers rather than being pure imaginary numbers), respectively. We will further exploit the fact that the algebra of complex numbers is the same as the algebra of real numbers. We will refer to this as mathematical "homomorphism". Then, the corresponding Helmholtz equations are

$$\frac{d^2V}{dz^2} - ZYV = 0 \tag{11.2.5}$$

$$\frac{d^2I}{dz^2} - ZYI = 0 \tag{11.2.6}$$

or

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0 \tag{11.2.7}$$

$$\frac{d^2I}{dz^2} - \gamma^2 I = 0 \tag{11.2.8}$$

where $\gamma^2 = ZY$, or that one can also think of $\gamma^2 = -\beta^2$ by comparing with (11.1.18) and (11.1.19). Then the above is homomorphic to the lossless case except that now, β is a complex number, indicating that the field is decaying and oscillating as it propagates. As before, the above are second order one-dimensional Helmholtz equations where the general solutions are

$$V(z) = V_+ e^{-\gamma z} + V_- e^{\gamma z} \tag{11.2.9}$$

$$I(z) = I_+ e^{-\gamma z} + I_- e^{\gamma z} \tag{11.2.10}$$

and

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = j\beta \tag{11.2.11}$$

Here, $\beta = \beta' - j\beta''$ is now a complex number. In other words,

$$e^{-\gamma z} = e^{-j\beta' z - \beta'' z}$$

is an oscillatory and decaying wave. Or focusing on the voltage case,

$$V(z) = V_+ e^{-\beta'' z - j\beta' z} + V_- e^{\beta'' z + j\beta' z} \tag{11.2.12}$$

[7] A math term for "similar in math structure". The term is even used in computer science describing a emerging field of homomorphic computing.

Again, letting $V_\pm = |V_\pm| e^{j\phi_\pm}$, the above can be converted back to the time domain as

$$V(z,t) = \Re e\{V(z,\omega)e^{j\omega t}\} \tag{11.2.13}$$
$$= |V_+|e^{-\beta'' z}\cos(\omega t - \beta' z + \phi_+) + |V_-|e^{\beta'' z}\cos(\omega t + \beta' z + \phi_-) \tag{11.2.14}$$

The first term corresponds to a decaying wave moving to the right while the second term is also a decaying wave but moving to the left. When there is no loss, or $R = G = 0$, and from (11.2.11), we retrieve the lossless case where $\beta'' = 0$ and $\gamma = j\beta = j\omega\sqrt{LC}$.

Notice that for the lossy case, the characteristic impedance, which is the ratio of the voltage to the current for a one-way wave, can similarly be derived using homomorphism:

$$Z_0 = \frac{V_+}{I_+} = -\frac{V_-}{I_-} = \sqrt{\frac{L}{C}} = \sqrt{\frac{j\omega L}{j\omega C}} \rightarrow Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \tag{11.2.15}$$

The above Z_0 is manifestly a complex number. Here, Z_0 is the ratio of the phasors of the one-way traveling waves, and apparently, their current phasor and the voltage phasor will not be in phase for lossy transmission line.

In the absence of loss, the above again becomes

$$Z_0 = \sqrt{\frac{L}{C}} \tag{11.2.16}$$

the characteristic impedance for the lossless case previously derived.